Factorizations of the Orlik-Solomon Algebras(Combinatorial Theory and Related Topics)

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1 Introduction.

Let L be a finite geometric lattice with the top element $\hat{1}$ and the bottom element $\hat{0}$, and the rank function r. Let $r = r(\hat{1})$. The characteristic polynomial of L is defined by

$$\chi(L;t) = \sum_{X \in L} \mu(\hat{0}, X) t^{r-r(X)}.$$

In the right handside μ is the Möbius function [6]. For certain geometric lattices including the supersolvable lattices [7], it is known that the characteristic polynomial $\chi(L;t)$ factors as

$$\chi(L;t) = \prod_{i=1}^{r}(t - d_i) \quad \text{(each } d_i \text{ is a nonnegative integer).}$$

In this paper we prove a sufficient condition (2.9) of the factorization of this type. The condition is stated as the existence of a "nice" partition of the set $\mathcal{A} = \mathcal{A}(L)$ of atoms of L. It is not difficult to check that a supersolvable geometric lattice admits a "nice" partition (2.4) .

In fact we will actually show a stronger result. Let us briefly explain about it. Let K be an arbitrary field. In [4, p.171] the Orlik-Solomon algebra $OS(L)$ of L over K was introduced. It is a graded anticommutative K-algebra. One of the most important results concerning $OS(L)$ is [4] :

$$\mathrm{Poin}(OS(L);t) = \sum_{X \in L} \mu(\hat{0}, X)(-t)^{r(X)}.$$

Here the left handside stands for the Poincaré series of the graded algebra $OS(L)$. Suppose that we have a partition $(\pi_1, \ldots, \pi_s)$ of the set $\mathcal{A}$ of atoms of L. Define

$$(\pi_i) := \text{ the vector space over K spanned by } 1 \text{ and the elemenets of } \pi_i$$

for $i = 1, 2, \ldots, s$.

Then the main theorem (2.8) in this paper is that there exists a natural graded vector space isomorphism

$$\kappa : (\pi_1) \otimes (\pi_2) \otimes \cdots \otimes (\pi_s) \to OS(L)$$

if and only if the partition $(\pi_1, \ldots, \pi_s)$ is "nice".

The above-mentioned sufficient condition easily follows from the main theorem.

2 Main Theorem and Its Corollaries.

Let $L, \mathrm{K}, \mathcal{A} = \mathcal{A}(L), OS(L)$ be as in the previous section.

Definition 2.1 A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called **independent** if atoms $H_1, \ldots, H_s$ are independent (i. e., $r(H_1 \vee \cdots \vee H_s) = s$) whenever $H_i \in \pi_i$ $(i = 1, \ldots, s)$.

For $X \in L$, define

$$L_X := \{Y \in L \mid Y \leq X\}, \quad \mathcal{A}_X := \mathcal{A}(L_X) = \{H \in \mathcal{A} \mid H \leq X\}.$$

Definition 2.2 Let $X \in L$. Let $\pi = (\pi_1, \ldots, \pi_s)$ be a partition of $\mathcal{A}$. Then the **induced partition** π_X is a partition of $\mathcal{A}_X$ whose blocks are the subsets $\pi_i \cap \mathcal{A}_X$ $(i = 1, \ldots, s)$ which are not empty.

Definition 2.3 A partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ is called **nice** if:

1) it is independent, and

2) the induced partition π_X contains a block which is a singleton unless $\mathcal{A}_X \neq \emptyset$.

Remark. In [2], M. Falk anf M. Jambu studied a similar partition. A major difference from ours lies in their assumption that the characteristic polynomial of L factors completely in $\mathbf{Z}[t]$.

Example 2.4 Let L be a supersolvable lattice. Then the set $\mathcal{A} = \mathcal{A}(L)$ admits a nice partition. In fact, define

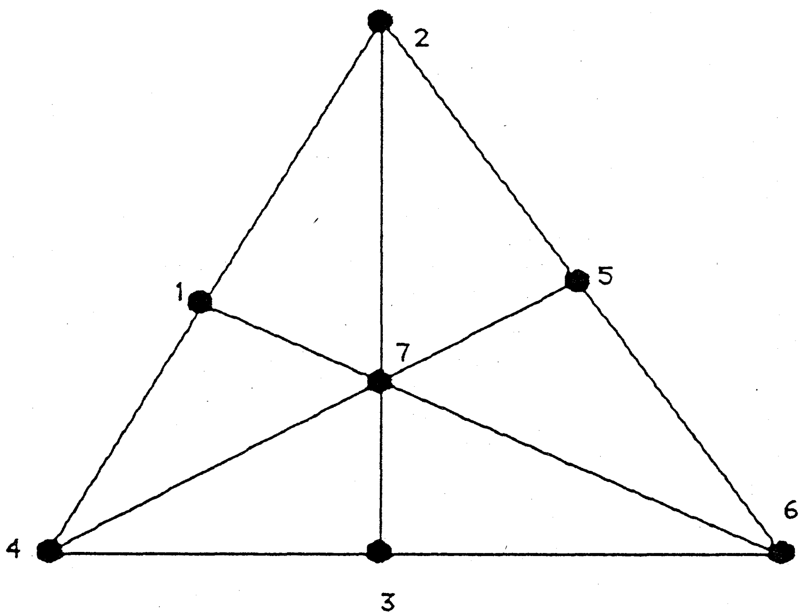
$$\pi_i = \{H \in \mathcal{A} \mid a \leq X_i; H \not\leq X_{i-1}\}$$

for a chain of modular elements

$$\hat{0} = X_0 < X_1 < \cdots < X_r = \hat{1} \quad (r(X_i) = i).$$

Then it is not difficult to show that a partition $\pi = (\pi_1, \ldots, \pi_r)$ is a nice partition.

Example 2.5 Consider the lattice arising from the following matroid (the non-Fano matroid)



For this, $\{\{1\}, \{2, 3, 4\}, \{5, 6, 7\}\}$ is a nice partition.

For a partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$, define a graded vector space

$$(\pi) := (\pi_1) \otimes (\pi_2) \otimes \cdots \otimes (\pi_s),$$

where each graded vector space (π_i) is as in the Introduction. Agree that $(\pi) = \mathrm{K}$ when $\mathcal{A} = \emptyset$. Since the Poincaré series $\mathrm{Poin}((\pi_i); t)$ of each (π_i) is equal to $(1 + |\pi_i|\, t)$, we obtain

$$\mathrm{Poin}((\pi); t) = \prod_{i=1}^{s} (1 + |\pi_i|\, t).$$

Definition 2.6 A k-tuple $I = (H_1, \ldots, H_k)$ $(k \geq 0)$ of elements of $\mathcal{A}$ is called a **k-section** of π if

$$H_i \in \pi_{n(i)}\ (i = 1, \ldots, k), \quad 1 \leq n(1) < n(2) < \ldots < n(k) \leq s.$$

For a k-section $I = (H_1, \ldots, H_k)$, define p_I by

$$p_I := x_1 \otimes \cdots \otimes x_s \in (\pi).$$

Here

$$x_j = \begin{cases} H_i & \text{if } j = n(i) \\ 1 & \text{if } j \notin \{n(1), \ldots, n(k)\}. \end{cases}$$

Then p_I is homogeneous of degree k. The graded K-vector space (π) has a basis $\{p_I \mid I \text{ is a section of } \pi\}$.

For the Orlik-Solomon algebra we keep the notation in [5]: For a k-tuple $I = (H_1, \ldots, H_k)$ $(k \geq 0)$ of atoms, the notation $a_I \in OS(L)$ stands for the class of the exterior product $e_{H_1} \wedge \ldots \wedge e_{H_k}$. Recall that each element of the Orlik-Solomon algebra $OS(L)$ can be (not necessarily uniquely) expressed as a linear combination of $\{a_I \mid I \text{ is a tuple of atoms}\}$.

Definition 2.7 Define

$$\kappa : (\pi) \longrightarrow OS(L)$$

as the homogeneous K-linear map of degree zero satisfying

$$\kappa(p_I) = a_I$$

for each section I of π.

The main theorem is:

Theorem 2.8 *The map κ is an isomorphism (as graded vector spaces) if and only if the partition π is nice.*

We will prove this theorem in the next section.

Corollary 2.9 *If there exists a nice partition $\pi = (\pi_1, \ldots, \pi_s)$, we have $s = r$ and*

$$\chi(L;t) = \sum_{X \in L} \mu(\hat{0}, X) t^{r - r(X)} = \prod_{i=1}^{r} (t - |\pi_i|).$$

Corollary 2.10 *If π is a nice partition, then the multiset $\{|\pi_1|, \ldots, |\pi_s|\}$ depends only upon L.*

Corollary 2.11 *If π is a nice partition, then*

$$r(X) = |\{i \mid \pi_i \cap \mathcal{A}_X \neq \emptyset\}|$$

for all $X \in L$.

Corollary 2.12 *Let $\mathcal{A}$ be an arrangement of hyperplanes in a vector space. Let L be the intersection lattice of $\mathcal{A}$. Suppose that there exists a partition $\pi = (\pi_1, \ldots, \pi_s)$ of $\mathcal{A}$ such that*

1) codim $(H_1 \cap \cdots \cap H_s) = s$ *whenever $H_i \in \pi_i$ $(i = 1, \ldots, s)$, and*

2) For every $X \in L$, there exists a block π_{i_X} of π such that the set $\{H \in \pi_{i_X} \mid X \subseteq H\}$ is a singleton.

Then $s = r(L)$ and

$$\chi(L;t) = \prod_{i=1}^{s} (t - |\pi_i|).$$

These corollaries, except 2.11 which will be proved in the next section, are immediate consequences from the main theorem.

3 Proof of Main Theorem

We keep the notation in the previous section. First we will review three results concerning the Orlik-Solomon algebra. Denote the homogeneous part of degree d of the graded algebra $OS(L)$ by $OS_k(L)$:

$$OS(L) = \bigoplus_{k=0}^{r} OS_k(L).$$

For a tuple $I = (H_1, \ldots, H_k)$ of atoms, let

$$\bigvee I = H_1 \vee \cdots \vee H_k \in L.$$

For each $X \in L$, define a vector subspace $OS_X(L)$ of $OS(L)$ which is generated by $\{a_I \mid \bigvee I = X\}$. Agree that $OS_0(L) = OS_{\hat{0}}(L) = \mathrm{K}$.

Lemma 3.1 *([4, 2.11]) For each $k \geq 0$, we have*

$$OS_k(L) = \bigoplus_{\substack{X \in L \\ r(X)=k}} OS_X(L).$$

Lemma 3.2 *([3, 1.7]) For $X, Y \in L$ with $Y \leq X$, there exists a natural isomorphism*

$$OS_Y(L_X) \xrightarrow{\sim} OS_Y(L).$$

Define a boundary map

$$\partial : OS_k(L) \longrightarrow OS_{k-1}(L) \quad (k = 1, \ldots, r)$$

to be the K-linear map satisfying

$$\partial(a_I) = \sum_{j=1}^{k} (-1)^{j-1} a_{I_j}$$

for any k-tuple $I = (H_1, \ldots, H_k)$ of atoms. Here

$$I_j = (H_1, \ldots, H_{j-1}, H_{j+1}, \ldots, H_k)$$

for $1 \leq j \leq k$.

Lemma 3.3 *([4, 2.18]) The complex $(OS_{\bullet}(L), \partial)$ is acyclic.*

Next let $\pi = (\pi_1, \ldots, \pi_s)$ be a partition of the set $\mathcal{A} = \mathcal{A}(L)$. We study the graded vector space (π). Denote the homogeneous part of degree k of (π) by $(\pi)_k$:

$$(\pi) = \bigoplus_{k=0}^{s} (\pi)_k.$$

For each $X \in L$, define a vector subspace $(\pi)_X$ of (π) which has a basis $\{p_I \mid I \text{ is a section with } \bigvee I = X\}$. Agree that $(\pi)_0 = (\pi)_{\hat{0}} = \mathrm{K}$.

Lemma 3.4 *Suppose that π is an independent partition. For each $k \geq 0$, we have*

$$(\pi)_k = \bigoplus_{\substack{X \in L \\ r(X)=k}} (\pi)_X.$$

Proof. By definition, the right handside is actually a direct sum. Note that $(\pi)_k$ has a basis

$$\{p_I \mid I \text{ is a } k\text{-section of } \pi\}.$$

Put $X = \bigvee I$. Then $p_I \in (\pi)_X$. We have $r(X) = k$ because π is independent.
∎

Lemma 3.5 *For $X, Y \in L$ with $Y \leq X$, there exists a natural isomorphism*

$$(\pi_X)_Y \xrightarrow{\sim} (\pi)_Y.$$

Proof. If I is a section of π with $\bigvee I = Y$, then $I \subseteq \mathcal{A}_Y \subseteq \mathcal{A}_X$. Thus I is also a section of π_X. This shows:

$$\begin{aligned} &\{I \mid I \text{ is a section of } \pi \text{ with } \bigvee I = Y\} \\ = \ &\{I \mid I \text{ is a section of } \pi_X \text{ with } \bigvee I = Y\}. \end{aligned}$$

Therefore an isomorphism

$$p_I \in (\pi_X)_Y \longmapsto p_I \in (\pi)_Y$$

is obtained by inserting "$1\otimes$" $r - r(X)$ times. ∎

Define a K-linear map

$$\partial : (\pi)_k \longrightarrow (\pi)_{k-1} \quad (k = 1, \ldots, s)$$

satisfying

$$\partial(p_I) = \sum_{i=1}^{k} (-1)^{j-1} p_{I_j}$$

for any k-section I of π . Then it is easy to check $\partial \circ \partial = 0$.

Lemma 3.6 *Suppose that a partition π of $\mathcal{A}$ contains a block which is a singleton. Then the complex $((\pi)_*, \partial)$ is acyclic.*

Proof. We can assume that π_1 is a singleton: $\pi_1 = \{a_1\}$. Suppose that $x \in (\pi)_k$ is a cycle: $\partial x = 0$. Write x as

$$x = a_1 \otimes x_1 + 1 \otimes x_2,$$

where $x_1, x_2 \in (\pi_2) \otimes \cdots \otimes (\pi_s)$. Then

$$0 = \partial x = 1 \otimes x_1 - a_1 \otimes (\partial x_1) + 1 \otimes (\partial x_2) = 1 \otimes (x_1 + \partial x_2) - a_1 \otimes (\partial x_1).$$

This imlies

$$x_1 = -\partial x_2.$$

Define

$$y = a_1 \otimes x_2 \in (\pi)_{k+1}.$$

Then

$$\partial y = 1 \otimes x_2 - a_1 \otimes (\partial x_2) = 1 \otimes x_2 + a_1 \otimes x_1 = x. \ ∎$$

Proof of Main Theorem.

Sufficiency:

Assume that π is a nice partition. We will prove by induction on $r(L) = r(\hat{1})$. When $r(L) = 0$, $\mathcal{A} = \emptyset$. Thus $(\pi) = \mathrm{K} = OS(L)$.

Assume that $r = r(L) > 0$. Note $s \leq r$ because π is independent. Consider a diagram

$$\begin{array}{ccccccccccccc} 0 & \rightarrow & (\pi)_r & \xrightarrow{\partial} & (\pi)_{r-1} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & (\pi)_1 & \xrightarrow{\partial} & (\pi)_0 & \rightarrow & 0 \\ & & \downarrow \kappa_r & & \downarrow \kappa_{r-1} & & & & \downarrow \kappa_1 & & \downarrow \kappa_0 & & \\ 0 & \rightarrow & OS_r(L) & \xrightarrow{\partial} & OS_{r-1}(L) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & OS_1(L) & \xrightarrow{\partial} & OS_0(L) & \rightarrow & 0. \end{array}$$

Here all of the vertical maps are induced from $\kappa : (\pi) \rightarrow OS(L)$. The top row is exact because of 3.6. The bottom row is exact because of 3.3. Note that

$$(\pi)_k = \bigoplus_{\substack{Y \in L \\ r(Y)=k}} (\pi)_Y \simeq \bigoplus_{\substack{Y \in L \\ r(Y)=k}} (\pi_Y)_Y$$

by 3.4 and 3.5. Also note that

$$OS_k(L) = \bigoplus_{\substack{Y \in L \\ r(Y)=k}} OS_Y(L) \simeq \bigoplus_{\substack{Y \in L \\ r(Y)=k}} OS_Y(L_Y)$$

by 3.1 and 3.2. By applying the induction assumption to L_Y for $r(Y) < r$, we know that κ_i $(i = 1, \ldots, r-1)$ are isomorphisms. Therefore κ_r is also an isomorphism. Putting these together, we get an isomorphism

$$\kappa : (\pi) \xrightarrow{\sim} OS(L).$$

Necessity:

Suppose κ is an isomorphism. First we will show that π is independent. Let I be a section of π . Then $p_I \neq 0$. So

$$a_I = \kappa(p_I) \neq 0.$$

This shows that I is indepenedent.

Next we will show that π_X contains a block which is a singleton unless $X = \hat{0}$. Since

$$(\pi) = \bigoplus_{Y \in L} (\pi)_Y, \quad OS(L) = \bigoplus_{Y \in L} OS_Y(L),$$

κ induces isomorphisms

$$(\pi)_Y \xrightarrow{\sim} OS_Y(L).$$

By 3.5 and 3.2, we obtain

$$\begin{aligned}(\pi_X) = \bigoplus_{Y \in L_X} (\pi_X)_Y \simeq \bigoplus_{\substack{Y \in L \\ Y \leq X}} (\pi)_Y \simeq \bigoplus_{\substack{Y \in L \\ Y \leq X}} OS_Y(L) &\simeq \bigoplus_{Y \in L_X} OS_Y(L_X) \\ &= OS(L_X).\end{aligned}$$

Let $X \neq \hat{0}$. Then

$$0 = \sum_{\substack{Y \in L \\ Y \leq X}} \mu(\hat{0}, Y) = \mathrm{Poin}(OS(L_X); 1) = \mathrm{Poin}((\pi_X); 1) = \prod_i (1 - |\pi_i \cap \mathcal{A}_X|).$$

This implies that π_X contains a block which is a singleton. ∎

Remark. In [1] A. Björner and G. Ziegler gave a sufficient condition for the map κ to be an isomorphism. The condition is the existence of a rooting map ρ for which the root complex $RC(L, \rho)$ factors completely. We do not know if the existence of a nice partition is enough to construct such a rooting map.

Proof of Corollary 2.11. As we saw in the proof of Main Theorem, the isomorphism κ induces isomorphisms

$$\kappa_X : (\pi_X) \xrightarrow{\sim} OS(L_X)$$

for all $X \in L$. So π_X is a nice partition of $\mathcal{A}_X$. By 2.9, we have

$$r(X) = r(L_X) = |\pi_X| = |\{i \mid \pi_i \cap \mathcal{A}_X \neq \emptyset\}|. \quad ∎$$

Since we have the factorization theorem for free arrangements [8], it is natural to pose

Problem. If an arrangement admits a nice partition, then is it free?

The converse is not true in general. (For example, the Coxeter arrangemnt D_4 has no nice partitions.)

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